THE SCREW AS A UNIT IN A GRASSMANNIAN SYSTEM OF THE SIXTH ORDER.

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An article appeared in the *Monatshefte für Math. und Physik*, *Jahrgang* II, 1891, *Hefte* 8 *und* 9, by E. Müller, entitled "*Die Liniengeometrie nach den Principien der Grassmannischen Ausdehnungslehre.*" In this article the author treats the line geometry as a sixth order Grassmannian system, of which the fundamental units are six linear complexes, each of the fifth order; i. e. each unit consists of all the right lines which satisfy the condition

$$LS = 0$$

in which S is a *screw*, i. e. the sum of two fixed right lines, say $S = L_1 + L_2$, and L is a variable right line. This seems to be a very complicated unit, since it consists of ∞^3 plane pencils of rays, one passing through each point of space. I propose to take a single screw as the fundamental unit and thus to obtain a *screw* geometry of the sixth order. We shall regard all screws as given in the normal form

$$S = e\varepsilon + a \mid \varepsilon ,$$

e being a definite point, ε a definite vector, and a a scalar constant, i. e. the screw is expressed as the sum of a definite line and a line at $\infty \perp$ to it, the latter being equivalent to a *plane*-vector $\perp$ to ε. This conception is perfectly definite in every respect, and therefore preferable to the sum of two *finite* lines which may be replaced in an infinite number of ways by two other finite lines.

In this paper the screw, regarded as the *fundamental unit form* of the system of the sixth order, will be designated as a *monoid*, and represented by μ, μ', μ_1, etc.

The system of monoids dependent on any two fixed monoids, as

$$\mu = x_1\mu_1 + x_2\mu_2 ,$$

in which μ has all possible values obtained by giving different values to x_1 and x_2, will be called a *dyoid*, and represented by δ.

The system dependent on three monoids, as

$$\mu = \sum_1^3 x\mu ,$$

will be called a *trioid*, and designated by τ.

The system dependent on four monoids,

$$\mu = \sum_1^4 x\mu ,$$

will be called a tetroid **T**: and finally the system dependent on five monoids,

$$\mu = \sum_1^5 x\mu ,$$

will be called a pentoid, π.

The *dyoid* is a skew surface of the 3rd order whose generators all cut at right angles a right line which is an axis of symmetry of the surface, each generator being the axis of a screw of definite pitch.

The *trioid* consists of one system of generators of an ∞ of skew conicoids 3 of which pass through each point of space. Hence 3 monoids of the system pass through each point of space. Also 3 monoidal axes lie in each plane in space.

The *tetroid* is a system such that through every point in space there passes a 2d order *cone* of monoids belonging to it, and in every plane in space there lies a parabola enveloped by monoidal axes belonging to it.

The *pentoid* is a system such that every straight line in space is an axis of one monoid belonging to the system and of only one.

Grassmann's complete system of combinatory multiplication may be applied, as he has shown, to any group of quantities which have a linear dependence upon each other, that is quantities connected by equations of the kind just written. Thus the circles in a given plane may be taken as elements; any four being taken as reference units, a fifth linearly connected with these may be any circle whatever in the plane, giving thus a system of the fourth order. Any circle dependent on two given circles has with each the same axis radical: any circle dependent on three given circles has with each of them the same centre radical.

Similarly, *spheres* may be taken as the units, when we have a system of the *fifth* order. These systems have been thoroughly treated by E. Müller in the Monatshefte für Math. und Physik, Jahrg. III–IV, 1892–93.

Since any screw whatever can be expressed linearly in terms of any six given screws, we have here a system of the sixth order.

Product of two monoids. The product of two screws, as considered in a paper read at the Cleveland meeting before Section A of the American Association for the Advancement of Science, we may, for distinction, designate as the *scalar* product; in it the screws were not regarded as fundamental units, but as dependent on points and vectors which were the fundamental units. The scalar product has the following mechanical interpretation. If S_1 be a

wrench acting upon a rigid body and producing a twist S_2, then S_1S_2 is the work done in the operation.

We define the combinatory product of two monoids, $\mu_1\mu_2$, as that portion of the dyoid fixed by the equation

$$\mu = x_1\mu_1 + x_2\mu_2$$

which is generated by μ in the following way:—1° make $x_1 = 1$ and let x_2 increase from 0 to 1 : 2° keep x_2 at unity and diminish x_1 from 1 to 0. If, as a special case, μ_1 and μ_2 become intersecting right lines, this makes μ generate the area of the parallelogram of which μ_1 and μ_2 are adjacent sides; and, in general, the idea is analogous, a curved surface limited by four planes, forming a prism of unlimited length, being substituted for the parallelogram. If $\mu_1 = e_1\varepsilon_1 + a_1 . | \varepsilon_1$ and $\mu_2 = e_2\varepsilon_2 + a_2 . | \varepsilon_2$, the prism has for one edge the common perpendicular on $e_1\varepsilon_1$ and $e_2\varepsilon_2$, and its sides are parallel to ε_1 and ε_2 and of a width equal to $T\varepsilon_1$ and $T\varepsilon_2$.

Condition that two double monoidal products shall be equal to each other, say

$$\mu_3\mu_4 = \mu_1\mu_2 \,.$$

Let

$$\mu_3 = x_1\mu_1 + x_2\mu_2 \,, \quad \mu_4 = y_1\mu_1 + y_2\mu_2 \,;$$

$$\therefore \qquad \mu_3\mu_4 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \mu_1\mu_2 \,.$$

Hence, the products of pairs of *monoids* belonging to the *same dyoid* can differ only by a *scalar multiplier*, which must be unity if the products are to be equal. Again, let

$$\mu_3 = x_1\mu_1 + y_1\mu'_1 \,, \quad \mu_4 = x_2\mu_2 + y_2\mu'_2 \,;$$

$$\therefore \qquad \mu_3\mu_4 = x_1x_2\mu_1\mu_2 + x_1y_2\mu_1\mu'_2 + y_1x_2\mu'_1\mu_2 + y_1y_2\mu'_1\mu'_2 \,.$$

This will evidently only reduce to a scalar multiple of $\mu_1\mu_2$ when μ'_1 and μ'_2 depend on μ_1 and μ_2 as in the previous case.

Hence, a double product of monoids can never be equal to another double product of monoids *unless they belong to the same dyoid.*

When $x_1y_2 - x_2y_1 = 0$, or $x_1/y_1 = x_2/y_2$, i. e., using the first value of $\mu_3\mu_4$, μ_4 differs from μ_3 only by a scalar multiplier, we have

$$\mu_3\mu_4 = 0 \,,$$

and in no other case.

Product of three monoids. The product $\mu_1\mu_2\mu_3$ is that portion of the trioid

$$\mu = \sum_1^3 x\mu$$

which is the locus of μ when x_1, x_2, x_3 vary through all positive values from 0 to 1. It can be shown, as in the case of double products, that no two triple products of monoids can be equal unless they belong to the same trioid, and any two which belong to the same trioid can differ only by a scalar multiplier.

Product of four monoids. The product $\mu_1\mu_2\mu_3\mu_4$ is that portion of the tetroid

$$\mu = \sum_1^4{}' x\mu$$

which is the locus of μ when $x_1, \ldots x_4$ vary through all positive values from 0 to 1. No two quadruple products can be equal unless they belong to the same tetroid, and any two belonging to the same tetroid can differ only by a scalar multiplier.

Product of five monoids. Precisely the same statement, *mutatis mutandis*, holds in this case as in the preceding.

Product of six monoids. This product will be always scalar.

If we have a continued product of *more* than six monoids, as a single combinatory product it must be zero, because the factors are not all independent, but it may also be interpreted by cutting off the six right hand factors as a scalar product by itself, then the next six, if that number are left, etc.

Any six monoids may be taken as a reference system in terms of which all other monoids may be expressed. We will designate the reference system by heavy figures **1, 2, 3, 4, 5, 6.**

The simplest systems that can be taken are 1°

$$e_0\iota_1 + |\,\iota_1\,, \quad e_0\iota_2 + |\,\iota_2\,, \quad e_0\iota_3 + |\,\iota_3\,,$$

$$e_0\iota_1 - |\,\iota_1\,, \quad e_0\iota_2 - |\,\iota_2\,, \quad e_0\iota_3 - |\,\iota_3\,;$$

2° The six edges of a tetrahedron, in which case the reference monoids are reduced to straight lines.

The fifteen double products of the reference units, **12, 13, 23,** etc., form a reference system to which all dyoids may be referred. A linear function of these double products will not, however, be, in general, a single dyoid, as will presently appear.

Similarly the twenty triple products of the reference units form a reference system for trioids, but a linear function of these will not, in general, be a single trioid.

The fifteen quadruple products of the reference units form a reference system for tetroids, with the same limitation as in the two preceding cases.

Finally, the six quintuple products form a reference system for pentoids, and any linear function of the six is a single definite pentoid.

The product of the six reference monoids is *scalar*, and will be taken as the scalar unit of screw space.

The number of possible screws in space is ∞^5, since there are ∞^4 right lines, and each of these may be the axis of an infinity of screws differing in pitch.

We will give to the word *complement* its regular Grassmannian signification, viz :—

The complement of a reference unit is the product of the other reference units so taken that the unit times its complement is positive unity.

The complement of a scalar is the scalar itself.

The complement of the sum of several quantities is the sum of the complements of the quantities.

The complement of the product of several quantities is the product of the complements of the quantities.

We thus find the following results :—

$$\begin{array}{lll}
\mid \mathbf{1} = \mathbf{23456} & \text{for} & \mathbf{1} \mid \mathbf{1} = \mathbf{123456} = \text{unity}, \\
-\mid \mathbf{2} = \mathbf{34561} & \text{“} & \mathbf{2} \mid \mathbf{2} = -\,\mathbf{234561} = \text{unity, etc.}; \\
\mid \mathbf{12} = \mathbf{3456} & \text{“} & \mathbf{12} \mid \mathbf{12} = \text{unity, etc.}; \\
\mid \mathbf{123} = \mathbf{456}, & \text{“} & \mid \mathbf{234} = -\,\mathbf{561}, \text{ etc.}; \\
\mid \mathbf{1234} = \mathbf{56}, \text{ etc.}; & & \mid \mathbf{12345} = \mathbf{6}, \text{ etc.};
\end{array}$$

$$\mid(\mid \mathbf{1}) = -\,\mathbf{1}, \quad \mid(\mid \mathbf{12}) = \mathbf{12}, \quad \mid(\mid \mathbf{123}) = -\,\mathbf{123},$$

$$\mid(\mid \mathbf{1234}) = \mathbf{1234}, \text{ etc.}$$

If we let $\mu_1 = x_1\,\mathbf{1} + x_2\,\mathbf{2} + \ldots + x_6\,\mathbf{6}$, and $\mu_2 = y_1\mathbf{1} + \ldots + y_6\,\mathbf{6}$, then

$$\mu_1\mu_2 = \left\Vert \begin{matrix} x_1, x_2, x_3, x_4, x_5, x_6 \\ y_1, y_2, y_3, y_4, y_5, y_6 \end{matrix} \right\Vert (\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}),$$

which signifies that $\mu_1\mu_2$ consists of the sum of all the determinants of the second order that can be formed out of the x's and y's, each multiplied into its corresponding double product of reference units.

A similar expression will give the product of a larger number of monoids.

Let $\mu_1, \ldots, \mu_6$ be any six monoids; the sum of their 15 double products, each multiplied by a scalar factor, may be written in the form

$$\mu_1 \sum_2^6 x\mu + \mu_2 \sum_3^6 y\mu + \mu_3 \sum_4^6 z\mu + u_4u_5u_6 \left[\frac{\mu_5}{u_5} - \frac{\mu_4}{u_4}\right]\left[\frac{\mu_6}{u_6} - \frac{\mu_4}{u_4}\right].$$

The sum thus appears as the sum of four dyoids, and cannot be further reduced, except for special values of the scalar coefficients. If any number of dyoids be added together, each may be expressed in terms of the 15 double products above, when the sum will appear as above, each coefficient being made up of the sum of the corresponding coefficients for each of the dyoids. It follows that, in general, the sum of any number of dyoids can be reduced to the sum of four dyoids. This can be done in an infinite number of ways, because *any* six monoids whatever can be taken for the reference system. Taking the complementary expression we have a similar statement for the sum of any number of tetroids. This points towards new Grassmannian systems of the 15th order having fundamental units each made up of the sum of four dyoids or tetroids.

Analogous results may be obtained with trioids.

Progressive and Regressive Products. We have a progressive product of two factors when the sum of the orders of the factors does not exceed 6, and regressive when it does. Thus $\mu\delta$, $\mu\tau$, $\mu\mathbf{T}$, $\mu\pi$, $\delta_1\delta_2$, $\delta\tau$, $\delta\mathbf{T}$, etc. are progressive products, while $\delta\pi$, $\tau\pi$, $\tau\mathbf{T}$, $\mathbf{T}\pi$, etc. are regressive. The progressive products are immediately interpretable; thus $\mu\delta$ is a trioid, $\mu\mathbf{T}$ a pentoid, $\delta_1\delta_2$ a tetroid, etc. In the case of the regressive products the product is the common figure multiplied by a scalar factor. Thus let $\delta = \mu_1\mu_2$ and $\pi = \mu_1\mu_3\mu_4\mu_5\mu_6$; then the common figure is the monoid μ_1, and the product is

$$\delta\pi = \mu_1\mu_2 \cdot \mu_1\mu_3\mu_4\mu_5\mu_6 = \mu_1\mu_2\mu_3\mu_4\mu_5\mu_6 \cdot \mu_1 .$$

If the common monoid is not given explicitly it may still be found in terms of any two monoids whose product is equivalent to δ or any five whose product is equivalent to π. Thus, let $\delta = \mu_1\mu_2$ and $\pi = \mu_3\mu_4\mu_5\mu_6\mu_7$; then the common monoid must belong to δ and therefore be expressible as a linear function of μ_1 and μ_2. Hence $\delta\pi = 12 \,.\, 34567 = x_1 1 + x_2 2$, say, using 1, 2, etc. instead of μ_1, μ_2, etc.

Multiply both sides of the equation into the pentoid 23456;

$$\therefore \qquad 12 \,.\, 34567 \,.\, 23456 = x_1 \,.\, 123456 ,$$

or

$$123456 \,.\, 345672 = x_1 \,.\, 123456 .$$

$$\therefore \qquad x_1 = -\, 234567 .$$

Similarly,

$$x_2 = -\, 345671 .$$

Hence

$$\delta\pi = -\, \mu_1 \cdot \mu_2\mu_3\mu_4\mu_5\mu_6\mu_7 - \mu_2 \cdot \mu_3\mu_4\mu_5\mu_6\mu_7\mu_1 .$$

Any case may be treated in a similar way.

In closing I will say that the discussion of this system is especially interesting to my mind, because it gives a definite and reasonably simple geometric conception of a sixth order space, the formulæ and operations being precisely what they would be in a discussion of six-dimensional vector space. In the latter case of course we could form no conception of the geometric meaning of our expressions, the operations would be simply *formal,* while in the case treated every expression has a meaning as definite and concrete as in ordinary three-dimensional space.